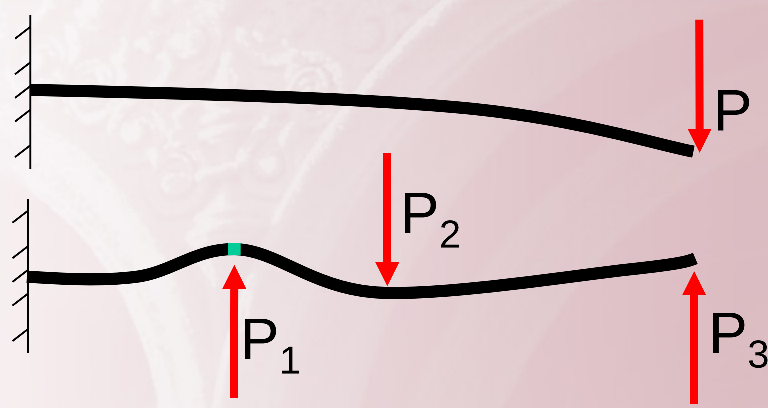
Structures – Supplemental Notes

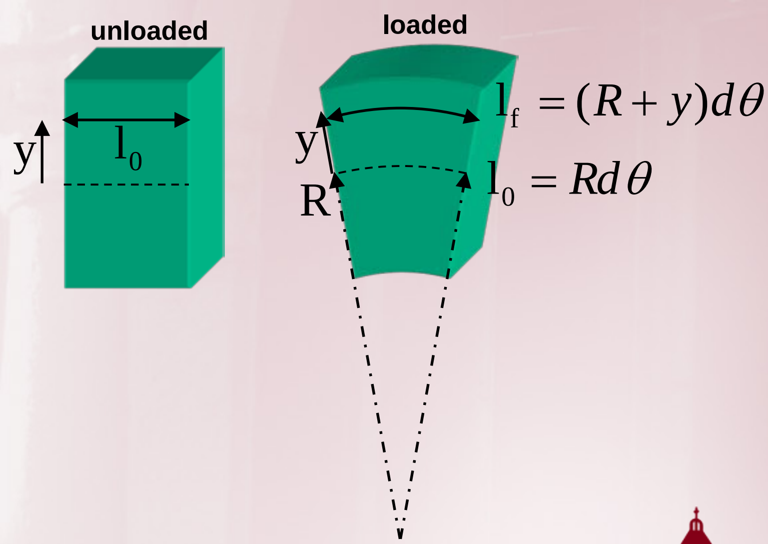
- Following Slides Derive Beam Bending Formulas Used in Hmwk, Tests
  - Blue boxes: formulas you will apply
  - Dashed blue boxes: intermediate results that will be applied later
  - Unboxed: steps you are not responsible for in this class
- General Results are Applied to a Canonical Cantilever Beam
- Final Slide Clarifies Symbols for Directions vs Deformations

Beam Bending – Strain Geometry

For any way a beam is being bent, for any number of external forces (P) ...



You can examine a small slice of it and make basic geometric observations

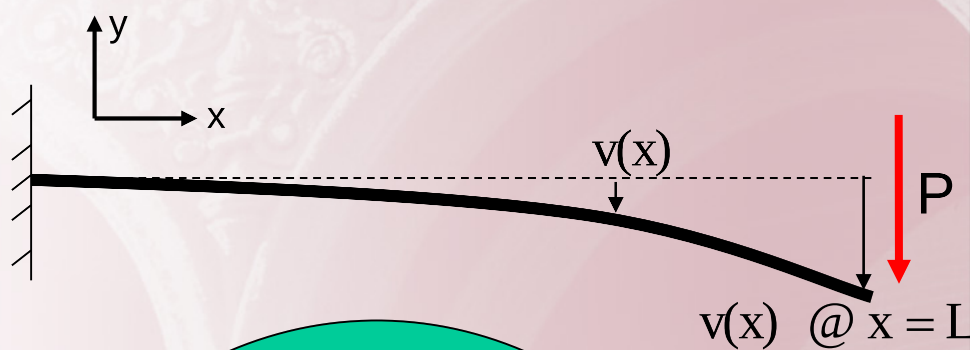


- There is an instantaneous radius of curvature (R)
- Initially, the slice has the same length everywhere
- After it deforms, the slice has the same length only at the center (the neutral axis). Above and below it becomes longer and shorter.
- Mathematically, we can find strain (ε) as a function of position in the beam (y) and radius of curvature (R).

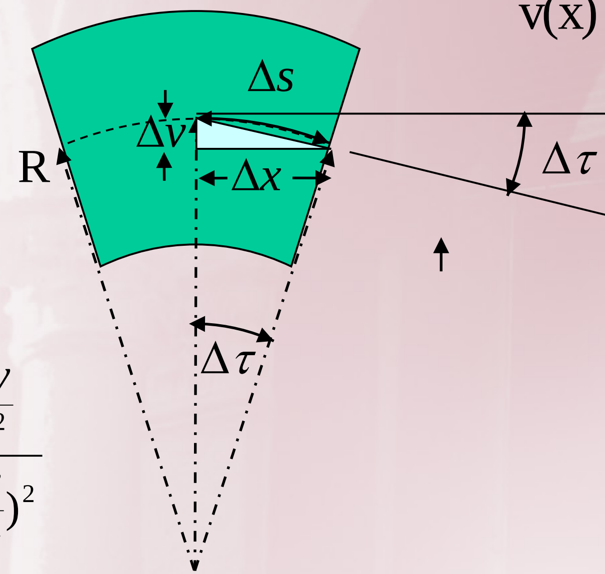
$$\varepsilon = \frac{\Delta l}{l_0} = \frac{l_f - l_0}{l_0} = \frac{(R+y)d\theta - Rd\theta}{Rd\theta} = \frac{y}{R}$$

Beam Bending – More Geometry

- Terrific, but knowing R at a particular point isn't a good way of describing the deformed beam shape.
- Instead we want to know the deformation v in the y direction as a function of x position on the beam.



With math tricks that **you don't need to know for this class**, you can relate v to R



- From the length of a circle segment we can get

$$R \cdot \Delta\tau = \Delta s \Rightarrow \frac{1}{R} = \frac{\Delta\tau}{\Delta s} = \frac{d\tau}{ds}$$

- The slope of the neutral axis can be described and manipulated to get

$$\tan(\Delta\tau) = -\frac{\Delta v}{\Delta x} = -\frac{dv}{dx} \Rightarrow \Delta\tau = -\arctan\left(\frac{dv}{dx}\right) \Rightarrow \frac{d\tau}{dx} = \frac{-\frac{d^2v}{dx^2}}{1+\left(\frac{dv}{dx}\right)^2}$$

- Finally, assuming differentially small lengths...

$$ds^2 = dy^2 + dx^2 \Rightarrow \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \Rightarrow \boxed{\frac{1}{R}} = \frac{d\tau}{ds} = \frac{d\tau}{dx}\frac{dx}{ds} = \frac{-\frac{d^2v}{dx^2}}{\left(1+\left(\frac{dv}{dx}\right)^2\right)^{\frac{3}{2}}} = \boxed{-\frac{d^2v}{dx^2}}$$

higher order terms ≈ 0 for small angles

Beam Bending – Stress and Forces

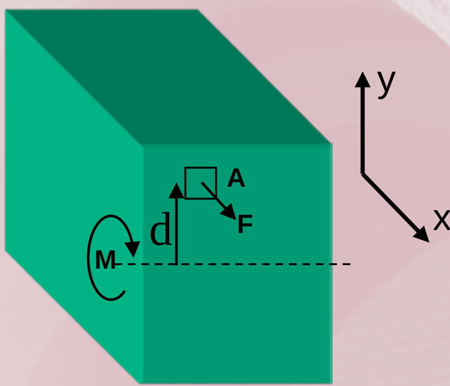
- If there is stress (σ) acting in the cross-section, it causes a moment around the neutral axis of the beam

A = Area over which stress acts
d = distance of force from axis beam is bending around

$$M = Fd = \sigma A d$$

- The integral of the moment from stress over the whole area of the cross-section is the total moment acting at the x location where the cross-section was sliced

$$M_{beam} = \int_A \sigma dA y$$



Using our previous results we relate moment and beam deflection (v)

$$M_{beam} = \int_A \varepsilon E y dA$$ • Stress/Strain relationship $\sigma = \varepsilon E$ (Young's Modulus / Elastic modulus)

$$M_{beam} = \int_A \frac{y}{R} E y dA$$ • Strain from curved geometry $\varepsilon = \frac{y}{R}$

$$M_{beam} = -\int_A y \frac{d^2 v}{dx^2} E y dA$$ • Deflection geometry $\frac{1}{R} = -\frac{d^2 v}{dx^2}$

$$M_{beam} = -\frac{d^2 v}{dx^2} E \int_A y^2 dA$$ • Constants out of the integral

$$M_{beam} = -\frac{d^2 v}{dx^2} EI$$ • Define moment of inertia $I = \int_A y^2 dA$

And we can manipulate this to yield

$$\sigma = -\frac{M_{beam} y}{I}$$

A very useful way of assessing stress if we know the internal moment of the beam and the moment of inertia

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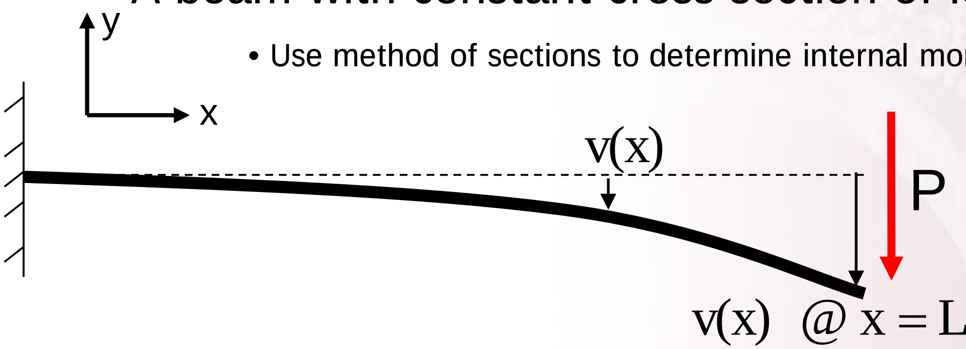
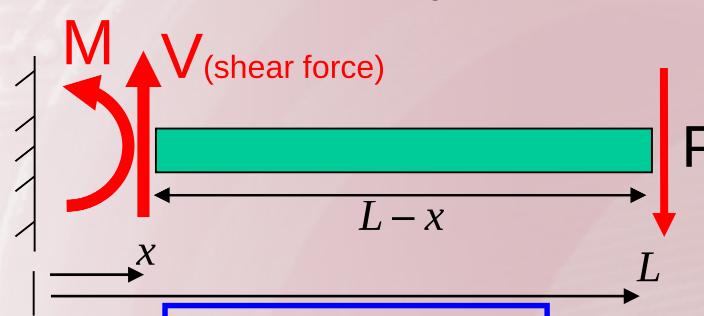
Beam Bending – Cantilever Beam Example

A beam with constant cross section of length L with load P at one end

- Use method of sections to determine internal moment as a function of distance x along beam





$$M_{beam}(x) = P(L-x)$$

- Recast to get deformation v as a function of moment and ultimately load

$$M_{beam} = -\frac{d^2v}{dx^2}EI \qquad \frac{d^2v}{dx^2} = -\frac{M_{beam}}{EI} = -\frac{P(L-x)}{EI}$$

- Integrate to get slope of deformation

$$\frac{dv}{dx} = \int_0^x \frac{d^2v}{dx^2} = -\frac{P}{EI}\int_0^x (L-x) = -\frac{P}{EI}\left[Lx - \frac{x^2}{2}\right] + C_1$$

Slope at x = 0 is 0 (fixed cantilever boundary condition) so C1 = 0

- Integrate to get deformation

$$v = \int_0^x \frac{dv}{dx} = -\frac{P}{EI}\int_0^x \left[Lx - \frac{x^2}{2}\right] = -\frac{P}{EI}\left[\frac{Lx^2}{2} - \frac{x^3}{6}\right] + C_2$$

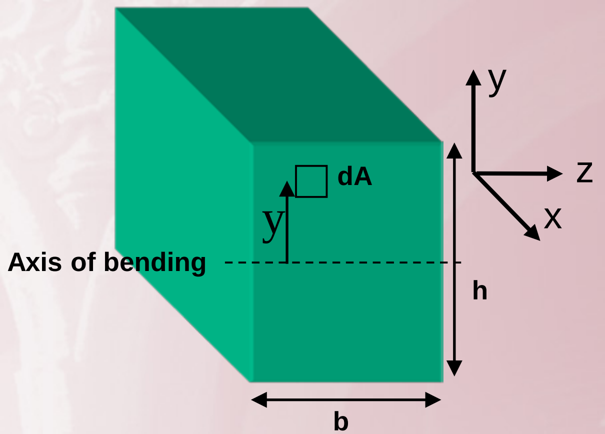
Displacement at x = 0 is 0 (fixed cantilever boundary condition) so C2 = 0

Maximum deformation occurs at the end of the beam and is

$$v_{x=L} = -\frac{PL^3}{3EI}$$

Moment of Inertia for a Rectangular Section

- **You <u>don't</u> need to know this derivation for this class**. In practice you can look up moment of inertia (I) for many cross-sections



  - Definition of moment of inertia for an arbitrary cross section

$$I = \int_A y^2 dA$$

  - Integral evaluated for a rectangular cross-section

$$I = \int_0^b \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 dy dx = \int_0^b \left. \frac{y^3}{3} \right|_{-\frac{h}{2}}^{\frac{h}{2}} dx = \int_0^b \left(\frac{h^3}{24} + \frac{h^3}{24} \right) dx = \int_0^b \frac{h^3}{12} dx = \frac{bh^3}{12}$$

- **You <u>do</u> need to know the implications of the result** and be able to apply them
  - The further material is away from the neutral axis of a beam, the more dramatically it increases the moment of inertia
  - Higher moment of inertia – higher stiffness, lower deformation, lower stress under the same load conditions

$$I = \frac{bh^3}{12}$$

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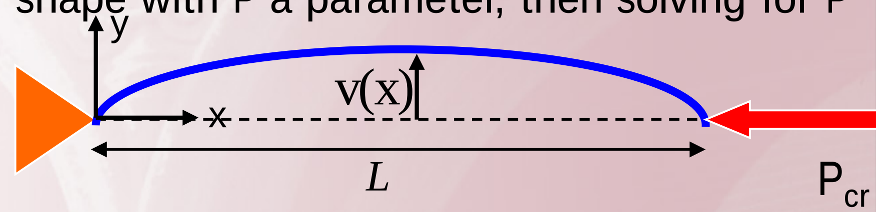
Failure – Euler Stability Criterion

- **Euler Buckling Formula –** Consider a beam under compression from a load P. At some critical load, the beam will buckle, taking on a deformed shape. Criteria derived from constitutive equations by assuming a deformed shape with P a parameter, then solving for P



- Moment at any location on beam is a function of load and moment arm v

$$M_{beam} = P_{critical} \cdot v(x)$$

- We also have an expression for moment in terms of deformation and beam properties

$$M_{beam} = -\frac{d^2 v}{dx^2} EI$$

- Equating them, we can develop a second order differential equation for v with a possible solution made of a sine and cosine

$$\frac{d^2 v}{dx^2} = -\frac{P_{critical}}{EI} v \qquad v = C_1 \sin\left(\sqrt{\frac{P_{critical}}{EI}} x\right) + C_2 \cos\left(\sqrt{\frac{P_{critical}}{EI}} x\right)$$

- For the example of the pinned-pinned beam above, apply the boundary conditions to determine constants

$$v(0) = 0 = C_1 \sin\left(\sqrt{\frac{P_{critical}}{EI}} 0\right) + C_2 \cos\left(\sqrt{\frac{P_{critical}}{EI}} 0\right) = C_2 \Rightarrow C_2 = 0$$

$$v(L) = 0 = C_1 \sin\left(\sqrt{\frac{P_{critical}}{EI}} L\right) \Rightarrow \sqrt{\frac{P_{critical}}{EI}} L = n\pi$$

- Rearrange to get critical load

$$P_{critical} = EI\left(\frac{n\pi}{L}\right)^2$$

- Other boundary conditions yield other criteria generalized with "effective length"

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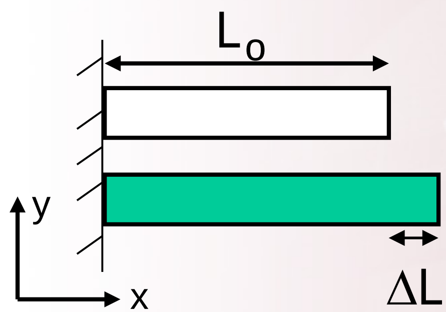
Terminology: Directions and Deformations

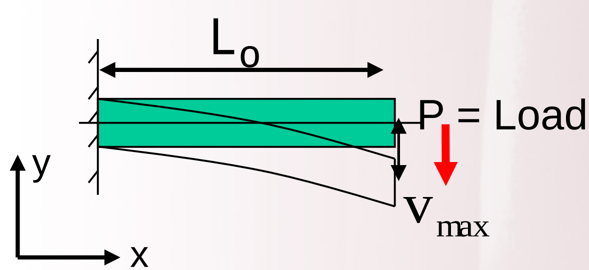
- Why are you using both x,y,z *and* u,v,w? That's confusing!
- *Directions* are in x,y,z, and never change
- *Deformations* in the x,y,z directions are described by u,v,w
- Examples:

$$\varepsilon = \frac{\Delta L}{L_o} = \frac{du}{dx}$$



The point we can describe as at $x = L_o$ on the undeformed beam moved $u = \Delta L$ in the x direction.

$$v_{max} = -\frac{PL^3}{3EI}$$



The point we can describe as at $x = L_o$ on the undeformed beam moved $v = v_{max}$ in the y direction

Statically Determinate and Indeterminate

- A structure is Statically Determinate when static equilibrium force equations (**Σ** moments = 0, **Σ** forces = 0) are sufficient to solve for all external reaction forces and internal forces, i.e. the number of static equilibrium equations you can formulate = the number of external reaction forces

- A structure is Statically Indeterminate when material properties and/or equality of deformations segments in the structure must be considered to determine all external reaction forces and internal forces